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Expanding gas clouds of ellipsoidal shape: new exact solutions

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We consider a model of a freely expanding gas cloud of tri-axial ellipsoidal shape, Gaussian density profile, proposed by Dyson (1968). The ellipsoids are deformable, but are further constrained here to have principal axes maintaining a fixed orientation in space: the study of the more general rotating flows is deferred to a future work. Our main result is that, when the fluid is a monatomic gas with adiabatic index $\gamma = 5/3$, the model is completely integrable by quadratures. Solutions starting from a state of rest are describable by elliptic functions; the generic solution however is a more general transcendent that cannot be reduced to elliptic type.

The complete integrability of Dyson's model may be ascribed to the fact that it possesses the Painlevé property (Ince 1956; Ablowitz & Segur 1977), meaning, essentially, that the solutions are meromorphic functions of the independent variable, admitting only pole singularities. However, the correct choice of independent variable here is not just the physical time t: rather, it is the *thermasy* (van Danzig 1939) $u = \int T\,\mathrm{d}t$, which is one of the potentials occurring in the Clebsch transformation.

Further investigation will be required to test Dyson's full 'spinning gas cloud' model for an eventual Painlevé property.

1. Introduction

A mass of incompressible fluid, under the influence of gravitation and pressure forces, adopts an ellipsoidal equilibrium shape: exact configurations were discussed first in detail by Dirichlet (1860); Dedekind (1860) and Riemann (1861); they found the most general solution possible, under the crucial assumption that the velocity field has a linear dependence on the spatial coordinates. A remarkable feature of this assumption is the existence of a particular symmetry, Dedekind's Duality Principle, according to which the vorticity and angular momentum vectors play symmetrical roles. From a group theoretic viewpoint, the ordinary O(3) rotational invariance of the Euler equations of motion gives rise, by the duality principle, to a new O(3) group which is its image, and thus to an enlarged symmetry group O(4), since $O(3) \times O(3)$ is isomorphic to O(4), the four-dimensional rotation group.

These intriguing features have attracted the attention of many researchers, who have sought to generalize the theory and to improve the understanding of the known results, such as Chandrasekhar (1969), Carter & Luminet (1985), and many others. It would certainly be interesting to have a generalization of the theory to the case of a compressible fluid, but that has proved to be difficult.

In a remarkable article, entitled 'Dynamics of a spinning gas cloud', Dyson (1968) has shown that ellipsoidally stratified compressible fluid configurations which preserve an ellipsoidal shape in the course of their evolution, are indeed possible provided that the force of gravity is neglected. The ellipsoids are tri-axial, of time-dependent shape and orientation, and the fluids motion combines expansion, rotation and vorticity in the most general case, which is described by an ordinary differential system of order 18. These results are still based on the assumption of a velocity field linear in the coordinates.

In the present work, we restrict ourselves to a consideration of the case without rotation, which is described by a sixth-order differential system. Our main result is that, when the fluid is an ideal monatomic gas, characterized by the adiabatic index:

$$\gamma = 5/3,$$

the equations constitute *a completely integrable mechanical system*, whose integration can be reduced to quadratures.

2. The model

We summarize here the main features and governing equations of Dyson's spinning gas cloud model. The central assumption is that of a linear relation between Eulerian coordinates x_i and Lagrangian coordinates a_i:

$$x_j = F_{jk}(t)\, a_k, \tag{2.1}$$

where $\boldsymbol{F}(t)$ is a time-dependent 3×3 matrix. Its meaning, in Dyson's words, is that 'the entire volume of gas is assumed to flow by a continuous affine transformation of the space. Straight lines in the fluid remain straight, but lengths and angles in general change with time'. A detailed description of the fluid's deformation is provided by the canonical decomposition of the matrix $\boldsymbol{F}$:

$$\boldsymbol{F} = \boldsymbol{O}_1 \boldsymbol{D} \boldsymbol{O}_2,$$

where $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ are orthogonal matrices and $\boldsymbol{D}$ is diagonal; to quote Dyson again: "the (above) representation always exists, and is in general ambiguous only to the extent of sign-changes and permutations of the three elements (D_1, D_2, D_3) of $\boldsymbol{D}$. Physically speaking, $\boldsymbol{O}_1$ defines the orientation of the gas with respect to the Eulerian coordinates $\boldsymbol{x}$, $\boldsymbol{O}_2$ defines the orientation with respect to the Lagrangian coordinates $\boldsymbol{a}$, and $\boldsymbol{D}$ defines the shape of the mass-distribution". ($\boldsymbol{O}_2$ determines in particular which fluid elements make up the principal axes.)

As a consequence of (2.1), the velocity components v_j are given by

$$v_j = \dot{F}_{jk}(t)\, a_k, \tag{2.2}$$

where the dot represents differentiation with respect to time. (Hence the linear dependence of $\boldsymbol{v}$ on $\boldsymbol{x}$, mentioned in the Introduction.)† The determinant of $\boldsymbol{F}$ represents the degree of expansion of the fluid, and is denoted $\phi(t)$:

$$\phi(t) \equiv \det(\boldsymbol{F}). \tag{2.3}$$

Differentiation with respect to Eulerian and Lagrangian coordinates is related by the chain rule:

$$\frac{\partial}{\partial a_k} = \frac{\partial x_j}{\partial a_k}\frac{\partial}{\partial x_j} = F_{jk}\frac{\partial}{\partial x_j} = (F_T)_{kj}\frac{\partial}{\partial x_j},$$

† The physical motivation for considering the simplifying assumption (2.1) or equivalently (2.2) on the form of the velocity field, lies essentially in the fact that the basic kinematical quantity, the deformation tensor $D_{ij} \equiv \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ is then uniformly distributed throughout space. That assumption was first introduced by Dirichlet, who applied it with great success to the problem of planetary figures of equilibrium, and it may be viewed as a natural generalization of the rigid flows that obtain when the uniform value of the deformation tensor vanishes.

where the subscript T denotes matrix transposition. Conversely Eulerian differentiation $\partial/\partial x_j$ is expressed by

$$\frac{\partial}{\partial x_j} = G_{jk}\frac{\partial}{\partial a_k}, \tag{2.4}$$

where
$$\boldsymbol{G} \equiv \boldsymbol{F}_T^{-1}. \tag{2.5}$$

Using (2.4), one obtains the velocity divergence in the form

$$\operatorname{div}\boldsymbol{v} = G_{jk}\dot{F}_{jk} = \operatorname{Tr}(\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}) = \dot{\phi}/\phi, \tag{2.6}$$

where Tr symbolizes the trace. Thus the continuity equation,

$$\operatorname{div}\boldsymbol{v} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\rho = 0 \tag{2.7}$$

(where $\mathrm{d}/\mathrm{d}t$ denotes the Lagrangian time derivative, following the element of fluid), can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\rho = -\dot{\phi}/\phi \tag{2.8}$$

or, in integrated form

$$\rho = \frac{f(\boldsymbol{a})}{\phi(t)} \tag{2.9}$$

($\boldsymbol{a} \equiv (a_1, a_2, a_3)$ is the Lagrangian position vector).

In his work, Dyson (1968) does not constrain the fluid to be polytropic, but merely to follow the perfect gas law:

$$U_{th} = U_{th}(T), \quad P = \rho T, \tag{2.10a, b}$$

where U_{th} is the specific internal energy, P the pressure, ρ the density, and T the absolute temperature, normalized in such a way that (2.10b) holds. The specific entropy S is then

$$S = -\ln\rho + \int\frac{\mathrm{d}U_{th}}{T}. \tag{2.11}$$

The fluid is assumed to evolve adiabatically starting from a state of uniform temperature distribution, which is then preserved in the course of the evolution. That this is so may be seen from the fact that the time evolution of temperature is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\frac{\mathrm{d}U_{th}}{T}\right) = \frac{-\dot{\phi}}{\phi} \tag{2.12}$$

(from $\mathrm{d}S/\mathrm{d}t = 0$), and is thus independent of location in space. (In the present work we shall assume the fluid to be a monatomic gas, characterized by an adiabatic index $\gamma = 5/3$, so that the entropy S is a function of $P/\rho^{5/3}$.)

The momentum equation:

$$\rho\dot{v}_j = -\frac{\partial P}{\partial x_j} \tag{2.13}$$

takes the form, taking account of equations (2.2), (2.4),

$$\rho\ddot{F}_{jk}a_k = -G_{jk}\frac{\partial P}{\partial a_k}. \tag{2.14}$$

In addition, using (2.10), (2.9) and the uniformity of the temperature distribution, we have

$$\frac{\partial P}{\partial a_k} = T\frac{\partial \rho}{\partial a_k} = P\frac{\partial \ln f}{\partial a_k} \tag{2.15}$$

so that the momentum equation (2.14) reads

$$\ddot{F}_{jk}\,a_k + TG_{jk}\frac{\partial \ln f}{\partial a_k} = 0.$$

That is $N(=3)$ equations on the coordinates a_k (N being the dimension of space), and that system should be identically satisfied in order for the model to be valid. $\partial \ln f/\partial a_k$ should then be linear in a_k (so that there may be cancellation with the linear $\ddot{F}_{jk}\,a_k$ terms), and therefore $\ln f$ should be quadratic. There is no loss of generality in redefining the Lagrangian coordinates in such a way that $\ln f = -\boldsymbol{a}^2/2$, i.e.

$$f(\boldsymbol{a}) = \text{constant} \times \exp(-\boldsymbol{a}^2/2). \tag{2.16}$$

(In view of the linear deformation relating Eulerian and Lagrangian coordinates, this describes an *ellipsoidally stratified fluid* – with three distinct axes in general.)

With the above Gaussian profile, the momentum equation becomes

$$(\ddot{F}_{jk} - TG_{jk})\,a_k = 0,$$

which is identically satisfied, as required, provided

$$\ddot{\boldsymbol{F}} = T\boldsymbol{G} \equiv T\boldsymbol{F}_T^{-1}. \tag{2.17}$$

The temperature T, or internal energy $U_{th}(T)$, is a given function of $\phi = \det(\boldsymbol{F})$ (see (2.12)), and therefore also a given function of the nine components F_{ij} of $\boldsymbol{F}$. Let us calculate its gradient in that space, $\partial U_{th}/\partial F_{jk}$: we have from (2.12) that $\mathrm{d}U_{th} = -T\,\mathrm{d}\ln\phi$, and, from $\phi \equiv \det(\boldsymbol{F})$:

$$\frac{\partial \ln \phi}{\partial F_{jk}} \equiv (\boldsymbol{F}_T^{-1})_{jk}$$

(since $\partial\phi/\partial F_{jk}$ is the cofactor associated with the matrix element F_{jk}). Dyson thus points out that the Euler equation (2.17) is of the form

$$\ddot{F}_{jk} + \frac{\partial U_{th}}{\partial F_{jk}} = 0, \tag{2.18}$$

which is the equation of motion of a point particle in nine-dimensional Euclidean space, under a force deriving from the potential U_{th}.

Dyson then discusses the constants of motion: there is first the *energy integral* immediately deducible from (2.18):

$$E = \tfrac{1}{2}\dot{F}_{jk}\,\dot{F}_{jk} + U_{th}. \tag{2.19}$$

There are the three components of *angular momentum*, which are here represented by the antisymmetric 3×3 matrix $\boldsymbol{J}$:

$$\boldsymbol{J} \equiv \boldsymbol{F}\dot{\boldsymbol{F}}_T - \dot{\boldsymbol{F}}\boldsymbol{F}_T. \tag{2.20}$$

By the Dedekind duality principle (which amounts to transposition of $\boldsymbol{F}$), the matrix $\boldsymbol{K}$ dual to $\boldsymbol{J}$ is also a constant of the motion:

$$\boldsymbol{K} = \boldsymbol{F}_T\dot{\boldsymbol{F}} - \dot{\boldsymbol{F}}_T\boldsymbol{F}; \tag{2.21}$$

these three new constants represent the components of *vorticity* relative to Lagrangian coordinate axes.

We will in the present work restrict ourselves to a consideration of tri-axial ellipsoids evolving without rotation, the three principal axes maintaining a fixed orientation in space; in such a case the matrix $\boldsymbol{F}$ must be diagonal, relative to these fixed coordinate axes:

$$\boldsymbol{F} = \begin{Bmatrix} a & & \\ & b & \\ & & c \end{Bmatrix}; \quad \phi = abc \tag{2.22}$$

(where a is not to be confused with a Lagrangian coordinate, a_i), and the Euler equation (2.17) takes the form

$$a\ddot{a} = b\ddot{b} = c\ddot{c} = \frac{\text{constant}}{(abc)^{\gamma-1}}, \tag{2.23}$$

a sixth-order differential system. (The constant in (2.23) is reducible by rescaling of either length or time, and will be taken to be unity, in what follows.) These are the equations that we propose to solve completely (by reduction to quadratures) in the monatomic case $\gamma = 5/3$.

3. Special symmetries of monatomic gas flow

3.1. *General results*

We now recall some special properties of the flow of a monatomic ideal gas, which turn out to be of crucial importance to the complete solvability of the mechanical system (2.23).

The essential point is the presence of a symmetry of the full set of gas-dynamical equations (continuity equation, momentum equation and energy equation, which is here the adiabatic condition $\mathrm{d}S/\mathrm{d}t = 0$) governing the most general three-dimensional flow of a monatomic gas, symmetry denoted (T^*) and characterized by the set of transformation formulae

$$t^* = 1/t; \quad \boldsymbol{x}^* = -\boldsymbol{x}/t; \quad \boldsymbol{v}^* = (\boldsymbol{v}t - \boldsymbol{x}); \quad \rho^* = \rho t^3; \quad P^* = Pt^5, \tag{3.1}$$

as was shown by Gaffet (1983).

By merely applying the symmetry, the law of conservation of energy E, which is, in divergence form

$$\operatorname{div}(P\boldsymbol{v}) + \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{v}^2}{2} + \frac{3P}{2\rho}\right) = 0 \tag{3.2}$$

gives rise to a new constant of the motion E^*, the image of E under the (reciprocal) transformation (T^*); by application to (3.2) of the transformation rules (3.1), we find that conservation of E^* is expressed by

$$\operatorname{div}[Pt(\boldsymbol{v}t - \boldsymbol{x})] + \rho\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(\boldsymbol{v}t - \boldsymbol{x})^2}{2} + \frac{3Pt^2}{2\rho}\right] = 0. \tag{3.3}$$

We remark that the density associated with E^*, by unit mass,

$$\frac{\delta E^*}{\delta M} = \frac{(\boldsymbol{v}t - \boldsymbol{x})^2}{2} + \frac{3Pt^2}{2\rho}$$

explicitly depends (quadratically) upon time. We have of course the freedom to perform an arbitrary translation of the origin of time, $t \to t+t_0$, and thus obtain a conserved current and density that is a second-degree polynomial in t_0. All three coefficients of this polynomial are necessarily conserved currents, two of them expressing conservation of E and E^*, while the third,

$$\operatorname{div}[P(\boldsymbol{x}-2\boldsymbol{v}t)]+\rho\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{v}t)-\frac{3Pt}{\rho}\right]=0, \tag{3.4}$$

expresses conservation of a new quantity, Σ say, with density

$$\frac{\delta\Sigma}{\delta M}=\boldsymbol{x}\cdot\boldsymbol{v}-t\left[v^2+\frac{3P}{\rho}\right]. \tag{3.5}$$

The constancy of Σ reflects a property of scale invariance of monatomic gas flow, and it may alternatively be derived through the Noether theorem.

If we combine these three densities we obtain: first,

$$\boldsymbol{x}\cdot\boldsymbol{v}=2t\frac{\delta E}{\delta M}+\frac{\delta\Sigma}{\delta M} \tag{3.6}$$

and then

$$\frac{\boldsymbol{x}^2}{2}=\frac{\delta E^*}{\delta M}+t\frac{\delta\Sigma}{\delta M}+t^2\frac{\delta E}{\delta M}. \tag{3.7}$$

This shows that *the polar moment of inertia I of an isolated mass of monatomic gas is a quadratic function of time*:

$$\frac{I}{2}\equiv\int\frac{\boldsymbol{x}^2}{2}\mathrm{d}M=E^*+t\Sigma+t^2E. \tag{3.8}$$

That property is intimately related to the virial theorem.

3.2. *Application to Dyson's model*

In the present case, where we have a Gaussian ellipsoidal density distribution with mean axes a, b, c, the polar moment of inertia I is easily determined to be

$$I=R^2\equiv a^2+b^2+c^2 \tag{3.9}$$

and the general results derived above (§3.1) show that

$$R^2/2=t^2E+t\Sigma+E^*, \tag{3.10}$$

where E^*, Σ and E are three constants of the motion. Differentiating twice we obtain, in sequence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2}{2}\right)=a\dot{a}+b\dot{b}+c\dot{c}=2Et+\Sigma, \tag{3.11}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{R^2}{2}\right)=(a\ddot{a}+b\ddot{b}+c\ddot{c})+(\dot{a}^2+\dot{b}^2+\dot{c}^2)=(\dot{a}^2+\dot{b}^2+\dot{c}^2)+\frac{3}{\phi^{2/3}}=2E. \tag{3.12}$$

Equations (3.10)–(3.12) determine the three constants of the motion E, Σ and E^* in terms of the dynamical variables a, b, c and their derivatives. In particular, we note the relation

$$\dot{R}^2=2E+(\Sigma^2-4EE^*)/R^2 \tag{3.13}$$

which may be viewed as an ordinary differential equation (o.d.e.) governing the time evolution of R; its solution is, of course, the quadratic formula (3.10). It is worth noting that *equation* (3.13) *has the form of the equation of motion for a free particle* of unit mass, energy E, and angular momentum $(4EE^*-\Sigma^2)^{1/2}$, in Euclidean space.

With the three first integrals E, Σ, E^* obtained, our sixth-order mechanical system (2.23) can in principle be reduced to one of the third order. (The remaining constants (2.20), (2.21) of angular momentum and vorticity identically vanish in the present case, and thus cannot be used to further decrease the order of the system).

3.3. *Equivalence with the dynamics of a point-mass on the 2-sphere*

From now on, $\boldsymbol{a} \equiv (a, b, c)$ denotes the three eigenvalues of the matrix $\boldsymbol{F}$ (i.e. the three principal axes of the ellipsoidal distribution), not a Lagrangian position vector. The equations of motion (2.23) manifestly constitute a Hamiltonian system in three-dimensional Euclidean space, and $\boldsymbol{a}$ may thus also be viewed as representing the position of a point particle in that space; that is the point of view that we will usually adopt in the remainder of this work. $R \equiv (a^2+b^2+c^2)^{1/2}$ is then the customary radial coordinate, the Euclidean distance to the origin.

The meaning of equation (3.13), or (3.10), is that *the problem of the determination of the radial motion $R(t)$ separates out*, leaving us with the simpler problem of determining the evolution of the two ratios

$$H \equiv b/a; \quad K \equiv c/a. \tag{3.14}$$

Clearly, H and K are related to the angular variables of a spherical coordinate system $(R, \bar{\theta}, \bar{\phi})$, defined by

$$a = R\cos\bar{\theta}, \quad b = R\sin\bar{\theta}\cos\bar{\phi}, \quad c = R\sin\bar{\theta}\sin\bar{\phi}, \tag{3.15}$$

namely, H and K may be identified with

$$H \equiv \tan\bar{\theta}\cos\bar{\phi}, \quad K \equiv \tan\bar{\theta}\sin\bar{\phi}. \tag{3.16}$$

Letting

$$\delta \equiv 1+H^2+K^2 \equiv \frac{1}{\cos^2\bar{\theta}} \tag{3.17}$$

the Euclidean coordinates a, b, c may be expressed, in terms of the new coordinates R, H, K, by

$$a = R/\delta^{1/2}, \quad b = aH = HR/\delta^{1/2}, \quad c = aK = KR/\delta^{1/2}. \tag{3.18}$$

We also have

$$\phi = a^3HK = R^3HK/\delta^{3/2}. \tag{3.19}$$

To obtain the equations of motion in this new coordinate system, it is convenient to start with the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{b}-b\dot{a}) = \frac{a/b-b/a}{\phi^{2/3}} \tag{3.20}$$

together with two more relations which can be deduced from it by circular permutation of a, b, c. Equation (3.20) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2}{\delta}\dot{H}\right) = \frac{1-H^2}{H\phi^{2/3}}, \tag{3.21}$$

while one of the other two relations obtained by permutation similarly yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2}{\delta}\dot{K}\right) = \frac{1-K^2}{K\phi^{2/3}}, \tag{3.22}$$

where ϕ is given by (3.19). We remark that R occurs merely through the combination $R^2\,\mathrm{d}/\mathrm{d}t$, in the above equations (3.21), (3.22), so that a simple and elegant way to eliminate R is just to *introduce a modified 'time' variable*:

$$t^* = \int \frac{\mathrm{d}t}{R^2(t)} \tag{3.23}$$

where $R(t)$ is the quadratic function of time (3.10).

Transformations of the type (3.23) have been discussed by various authors, and also appear in Gaffet (1983) (see references therein); the symmetry (T^*) considered in §3.1 may also be viewed as belonging to this category (letting: $R(t) = t$), hence the notation t^* for the new time-variable in (3.23).

With this redefinition of time, the equations of motion take the form

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t^*}\left(\frac{1}{\delta}\frac{\mathrm{d}H}{\mathrm{d}t^*}\right) &= \frac{\delta}{(HK)^{2/3}}\left(\frac{1}{H} - H\right), \\ \frac{\mathrm{d}}{\mathrm{d}t^*}\left(\frac{1}{\delta}\frac{\mathrm{d}K}{\mathrm{d}t^*}\right) &= \frac{\delta}{(HK)^{2/3}}\left(\frac{1}{K} - K\right). \end{aligned}\right\} \tag{3.24}$$

It is of course of interest to know whether this new system (3.24), is still Hamiltonian. In view of the symmetry of the problem and of the fact that H, K represent angular spherical coordinates, we would expect that if (3.24) is indeed a Hamiltonian system, it should describe the dynamics of a point on the surface of the unit sphere, in a potential. Let us then start with the Lagrangian:

$$L = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - V_S, \tag{3.25}$$

where $\mathrm{d}\sigma$ is the element of arclength on the unit sphere $x^2+y^2+z^2 = 1$, and V_S is the potential. We use coordinates $H \equiv y/x$, $K \equiv z/x$, and also introduce $\delta \equiv 1+H^2+K^2$ as before (note that $\delta \equiv 1/x^2$), and obtain

$$\mathrm{d}\sigma^2 = [\mathrm{d}H, \mathrm{d}K]\begin{bmatrix} (1+K^2); & -HK \\ -HK; & (1+H^2) \end{bmatrix}\begin{bmatrix} \mathrm{d}H \\ \mathrm{d}K \end{bmatrix}/\delta^2 = \frac{\mathrm{d}H^2+\mathrm{d}K^2+(H\,\mathrm{d}K-K\,\mathrm{d}H)^2}{\delta^2}. \tag{3.26}$$

The conjugate momenta are

$$\pi_1 \equiv \frac{\partial L}{\partial \dot{H}} = \frac{(1+K^2)\,\dot{H}-HK\dot{K}}{\delta^2}, \quad \pi_2 \equiv \frac{\partial L}{\partial \dot{K}} = \frac{(1+H^2)\,\dot{K}-KH\dot{H}}{\delta^2}. \tag{3.27}$$

The equations of motion are the Euler–Lagrange equations, and they assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{H}}}{\delta}\right) = \frac{-\boldsymbol{\nabla} V_S}{\delta}, \tag{3.28}$$

where $\dot{\boldsymbol{H}}$ denotes the two-vector of components $\dot{H}$, $\dot{K}$ and $\boldsymbol{\nabla}$ is the contravariant vector gradient, using the spherical metric as in (3.26). More explicitly, they are the equations

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{H}}}{\delta}\right) = \begin{bmatrix} (1+H^2)\dfrac{\partial V_S}{\partial H} + HK\dfrac{\partial V_S}{\partial K} \\ HK\dfrac{\partial V_S}{\partial H} + (1+K^2)\dfrac{\partial V_S}{\partial K} \end{bmatrix}. \tag{3.29}$$

In order that these coincide with the equations of our model, (3.24), we need only replace t by t^*, and choose a potential V_S defined by

$$\frac{\partial V_S}{\partial H} = \frac{3H^2-\delta}{H(HK)^{2/3}}; \quad \frac{\partial V_S}{\partial K} = \frac{3K^2-\delta}{K(HK)^{2/3}}. \tag{3.30}$$

This pair of equations is indeed compatible, and defines the potential

$$V_S = \frac{3\delta}{2(HK)^{2/3}} \equiv \frac{3/2}{(xyz)^{2/3}}. \tag{3.31}$$

This shows that equations (3.24) *do indeed describe the Hamiltonian motion of a point-mass on the* 2-*sphere*. In particular, we obtain the corresponding energy constant of the motion

$$\hat{E} = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t^*}\right)^2 + V_S, \tag{3.32}$$

where the kinetic term is given by (3.26), and the potential is as in (3.31). In fact, constancy of $\hat{E}$ is a direct consequence of the conservation of energy E of the original three-dimensional motion described by (2.23), as we now show.

The definition (3.12) of E may be rewritten

$$2E = \frac{\mathrm{d}s^2}{\mathrm{d}t^2} + \frac{3}{(abc)^{2/3}}, \tag{3.33}$$

where the three-dimensional line element $\mathrm{d}s^2$ may be decomposed into its radial and tangential parts:

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\sigma^2, \tag{3.34}$$

and where $(abc) \equiv R^3(xyz)$, x, y, z being the Cartesian coordinates of the trace of the radius vector on the unit sphere. Thus, (3.33) becomes

$$2E = \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + R^2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \frac{3}{R^2(xyz)^{2/3}},$$

and, if we substitute equation (3.13) for $\dot{R}^2$, and $R^2\,\mathrm{d}t^*$ for $\mathrm{d}t$ in agreement with the definition (3.23), we obtain

$$(4EE^* - \Sigma^2) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t^*}\right)^2 + \frac{3}{(xyz)^{2/3}}. \tag{3.35}$$

Thus we identify $2\hat{E}$ with $(4EE^* - \Sigma^2)$, the squared angular momentum that appears in equation (3.13).

4. The second integral, and the general solution of the system

The equations of motion (3.24) may be simplified further through consideration of a new independent variable u in place of t^*, the *thermasy*, introduced by van Danzig (1939), and which plays a central role in the Clebsch transformation of the velocity field in fluid dynamics (Seliger & Whitham 1968; see also Gaffet 1985; Carter & Gaffet 1988):

$$u = \int T\,\mathrm{d}t.$$

For a monatomic gas, $U_{th} = (P/\rho)/(\gamma-1) = T/(\gamma-1) = \frac{3}{2}T$ (with the choice of normalization (2.10*b*) that we have adopted for the temperature), and equation (2.12) integrates as

$$T\phi^{2/3} = \text{constant};$$

therefore:

$$u = \int \frac{\mathrm{d}t}{\phi^{2/3}} = \int \frac{\delta\,\mathrm{d}t^*}{(HK)^{2/3}}, \quad \text{where} \quad \delta \equiv 1+H^2+K^2. \tag{4.1}$$

The simplification lies essentially in the fact that the general solution will then be described, roughly speaking, by uniform (i.e. single-valued) functions of the independent variable u; more precisely, the differential system considered turns out to have the *Painlevé property* (as shown in Appendix A), when u is chosen as independent variable. According to a celebrated conjecture (Ablowitz & Segur 1977), that entails *complete integrability* of the system.

The Painlevé property holds, not exactly for the dependent variables H, K, but for the related variables U, V:†

$$U \equiv H^{2/3}, \quad V \equiv K^{2/3}.$$

The transformed system reads

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{U'}{VU^{1/2}}\right) = \frac{2}{3}\frac{1-U^3}{UU^{1/2}}, \tag{4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{V'}{UV^{1/2}}\right) = \frac{2}{3}\frac{1-V^3}{VV^{1/2}} \tag{4.2b}$$

(where a prime symbolizes derivation with respect to u), and it admits the first integral $\hat{E}$:

$$\tfrac{8}{9}\hat{E} = \left[\frac{(1+V^3)\,U'^2}{UV^2} - 2U'V' + \frac{(1+U^3)\,V'^2}{U^2V}\right] + \frac{4}{3UV}(1+U^3+V^3). \tag{4.3}$$

The system (4.2) presents some hidden symmetries which reflect its invariance under the group of permutation of the three principal axes a, b, c; this is discussed in Appendix B. We show in the Appendix that the hidden invariance properties of the system are made manifest through the consideration of a 3-vector $\boldsymbol{X}$, with components X, Y, Z, defined by (see (B 4))

$$\tfrac{3}{2}\boldsymbol{X} \equiv \frac{\boldsymbol{x} \wedge \boldsymbol{x}'(u)}{(xyz)^{2/3}},$$

where $\boldsymbol{x}$ is the Cartesian position vector on the surface of the unit sphere, introduced in §3.3 ($x \equiv a/R$, $y \equiv b/R$, $z \equiv c/R$, $\boldsymbol{x}^2 = 1$). The introduction of vector $\boldsymbol{X}$ simplifies considerably the formulation of several important results, such as that of the first integrals; its components X, Y, Z also occur naturally in the following useful reformulation of system (4.2):

$$\mathrm{d}u = \frac{\mathrm{d}U}{ZVU^{1/2}} = -\frac{\mathrm{d}V}{YUV^{1/2}} = -\frac{\mathrm{d}Z}{\frac{2}{3}\left(\frac{U^3-1}{U^{3/2}}\right)} = \frac{\mathrm{d}Y}{\frac{2}{3}\left(\frac{V^3-1}{V^{3/2}}\right)}. \tag{4.4}$$

† U and V are positive quantities, in order that H and K may be real numbers. H and K were originally introduced as positive quantities; however, their sign is relatively immaterial, as illustrated by the form of the equations of motion (3.24).

4.1. *The second integral, cubic in the momenta*

We expect, by application of the Painlevé conjecture, the existence of a second integral to this system. Second integrals are notoriously rare, and their search time-consuming, especially if they do not exist; the Painlevé property provides the motivation for the search. A new constant of the motion is indeed present, and it has the form

$$I_2 = \frac{3}{4}\left(\frac{U'V'}{UV}\right)\left(\frac{U'}{U}-\frac{V'}{V}\right)+\left[\frac{U'}{V^2}(V^3-1)-\frac{V'}{U^2}(U^3-1)\right]. \tag{4.5}$$

In terms of the 3-vectors $\boldsymbol{x}$ and $\boldsymbol{X}$, this can be written more compactly (and more symmetrically):

$$I_2 = \tfrac{3}{4}XYZ-(xyz)^{1/3}[X/x+Y/y+Z/z]. \tag{4.6}$$

The energy integral $\hat{E}$, in this notation, assumes the form

$$\tfrac{8}{9}\hat{E} = (X^2+Y^2+Z^2)+\tfrac{4}{3}/(xyz)^{2/3}. \tag{4.7}$$

X, Y, Z being linked by the relation (B 8) (Appendix B),

$$xX+yY+zZ = 0,$$

equations (4.6), (4.7) implicitly define X, Y, Z in terms of the position vector $\boldsymbol{x}$ and of the two integrals $\hat{E}$ and I_2; in other words, X, Y, Z become known functions of U and V, since the latter are just another way of parametrizing the 2-sphere $\boldsymbol{x}^2 = 1$.

Furthermore, we have, from (4.4),

$$\frac{dV}{dU} = -\frac{Y}{Z}\left(\frac{U}{V}\right)^{1/2}, \tag{4.8}$$

and that may be viewed as a first-order o.d.e. for the function $V(U)$, since Y and Z are now known functions of the coordinates U, V.

4.2. *The integral invariant, and the solution by quadrature*

We have succeeded in reducing our problem to the first-order o.d.e. (4.8) – where Y, Z are implicitly determined functions of U and V. To proceed further we remark that, in the form of (4.4), the differential system considered exhibits a fairly obvious *integral invariant* (see Goursat 1949), in the following way. Viewing u as the 'time', and (4.4) as representing motion in a four-dimensional space with coordinates $U^{1/2}$, $V^{1/2}$, Y, Z, the four velocity components are given by the denominators in (4.4), and it is clear without calculation that the velocity 4-vector is divergence-free. Thus the volume in that space:

$$\mathscr{V} \equiv \int\cdots\int dU^{1/2}\,dV^{1/2}\,dY\,dZ,$$

is conserved by the motion; that is the integral invariant. Such a property immediately entails integrability of (4.8) by quadrature (see Goursat 1949); in fact, (4.8) is now amenable to the form of an exact differential, $d\Phi$:

$$d\Phi = \frac{\partial(Y,Z)}{\partial(\hat{E},I_2)}(YU^{1/2}\,dU+ZV^{1/2}\,dV) \tag{4.9}$$

where the Jacobian is taken at U and V constant. In other words, the above Jacobian is an integrating factor of the first-order o.d.e. (4.8).

With this result, our task is in principle completed; in the following subsections we present some properties of the general solution, and also discuss some remarkable subsets of solutions.

4.3. *The solutions in the case where the second integral I_2 vanishes*

When $I_2 = 0$, the essential and surprising result is that V^2 and U^2 are linearly related as

$$V^2 = \alpha U^2 + \beta, \tag{4.10}$$

where α and β are two constants, algebraically related by the cubic equation

$$N(\alpha, \beta) \equiv \alpha^3 + \beta^3 + 3m\alpha\beta - 1 = 0, \tag{4.11}$$

where

$$m = \tfrac{2}{9}\hat{E}. \tag{4.12}$$

We remark that equations (4.10), (4.11) fully specify the trajectories of the point-mass on the unit sphere – since (U, V) is just a particular coordinate system on the sphere. Let us now give a derivation of these results.

First, we would like to simplify as far as possible the expression for the functions $Y(U, V)$, $Z(U, V)$ which appear in the o.d.e. (4.8), and which have been up to now only implicitly defined through the equations (4.6), (4.7) and (B 8). Let us introduce the ratio:

$$\alpha \equiv -\frac{Y}{Z}\left(\frac{V}{U}\right)^{1/2} \tag{4.13}$$

(later to be identified with the α occurring in (4.10)): the first-order equation (4.8) becomes

$$\frac{V\mathrm{d}V}{U\mathrm{d}U} = \alpha. \tag{4.14}$$

In terms of Z and α (instead of Y and Z), the definitions (4.6), (4.7) of the two integrals, $\hat{E} \equiv \frac{9}{2}m$ and I_2, read

$$\left(3m - \frac{\delta}{UV}\right) = \frac{3}{4}\frac{Z^2}{V}D, \tag{4.15}$$

where $\delta \equiv 1 + U^3 + V^3$ as usual, and

$$D \equiv \alpha^2 U(U^3+1) - 2\alpha U^2 V^2 + V(V^3+1), \tag{4.16}$$

$$\frac{I_2}{ZU^{1/2}} = \frac{3Z^2}{4V}\alpha(V^2 - \alpha U^2) + \left[\frac{V^3-1}{V} - \alpha\frac{U^3-1}{U}\right]. \tag{4.17}$$

The latter expression suggests introducing a new variable β defined by†

$$\beta = V^2 - \alpha U^2; \tag{4.18}$$

conversely, $V \equiv (\alpha U^2 + \beta)^{1/2}$ may be eliminated in favour of β. In practice, we will merely eliminate even powers of V (and odd powers of degree higher than unity),

† β is the image of α under permutation of a and b (see Appendix B); therefore, in view of definition (4; 13) of α, we must have $\beta \equiv -(X/Z)\,V^{1/2}$. Equation (4.18) then merely expresses the equation: $\boldsymbol{x}\cdot\boldsymbol{X} = 0$, and the definition (4.19) of D is simply: $D \equiv (V/Z^2)\boldsymbol{X}^2$.

retaining the symbol V as a convenient abbreviation for $(\alpha U^2+\beta)^{1/2}$; the definition (4.16) of D may thus be written

$$D \equiv \alpha^2 U + \beta^2 + V, \tag{4.19}$$

and the expressions (4.17) for the second integral becomes (using (4.15) to eliminate the Z^2 term)

$$\frac{I_2}{ZU^{1/2}} = \frac{N(\alpha,\beta)}{D(\alpha,\beta,U)}, \tag{4.20}$$

where $N(\alpha,\beta)$ is the cubic function of α and β defined by (4.11).

Let us now set $I_2 = 0$; we obtain

$$N(\alpha,\beta) = 0, \tag{4.21}$$

an encouragingly simple result. The above equation (4.21) defines a function $\beta(\alpha)$, and (4.18) may then be viewed as implicitly defining a function α (U, V), whose exact differential turns out to be

$$\tfrac{1}{2}\mathrm{d}\alpha = \frac{V\mathrm{d}V - \alpha U \mathrm{d}U}{U^2 - N_\alpha/N_\beta}, \tag{4.22}$$

where N_α, N_β are the partial derivatives $\partial N/\partial\alpha, \partial N/\partial\beta$. *Thus the general solution of the o.d.e.* (4.8), *or* (4.14), *is just*

$$\alpha(U, V) = \text{constant}, \tag{4.23}$$

the integrating factor being, $U^2 - N_\alpha/N_\beta$.

That completes the proof of (4.10), (4.11), since α has been identified with the integration constant, and β is then also constant as a consequence of (4.21).

Let us now examine the form of these solutions as a function of u, the independent variable. We need a rational parametrization of the trajectories (4.10), e.g.

$$U = \frac{\beta - \sigma^2}{2\sigma\alpha^{1/2}}, \quad V = \frac{\beta+\sigma^2}{2\sigma}, \tag{4.24}$$

where σ is the parameter representing an arbitrary point on the trajectory; we wish to determine the evolution of σ as a function of u. Using

$$\mathrm{d}U = -\frac{V}{\alpha^{1/2}}\mathrm{d}\ln\sigma, \quad \mathrm{d}V = -U\alpha^{1/2}\mathrm{d}\ln\sigma, \tag{4.25}$$

we obtain (see definition (4.4) of Z)

$$Z = \frac{-\sigma'(u)}{\sigma(\alpha U)^{1/2}}, \tag{4.26}$$

and then, substituting it in (4.17) where $I_2 = 0$, we find

$$\frac{3\beta}{4}\frac{\sigma'^2(u)}{\sigma^2} = \alpha V(U^3-1) - U(V^3-1) = \frac{\beta P_4(\sigma)}{4\sigma^2\alpha^{1/2}}, \tag{4.27}$$

where $P_4(\sigma)$ is the fourth-degree polynomial

$$P_4(\sigma) \equiv \sigma^4 - \frac{2\sigma^3}{\beta}(1+\alpha^{3/2}) + 2\sigma(1-\alpha^{3/2}) - \beta^2. \tag{4.28}$$

σ thus turns out to be the elliptic function of u defined by

$$\sigma'^2(u) = \frac{P_4(\sigma)}{3\alpha^{1/2}}. \tag{4.29}$$

The invariants (Goursat 1949) associated with that elliptic function are

$$g_2 = \tfrac{1}{3}m, \quad g_3 = \tfrac{1}{27}, \tag{4.30}$$

meaning that it is reducible to the Weierstrass canonical form

$$W'^2(u) = 4W^3 - \tfrac{1}{3}mW - \tfrac{1}{27}, \tag{4.31}$$

e.g. through a homographic (or Moebius) transformation. The transformation is explicitly given in Appendix C. It is remarkable that the invariants (4.30), and hence the Weierstrass function, *do not depend on the value of the integration constant* α.

For an elliptic function of the type (4.29), the associated Weierstrass function assumes the form

$$2W(u) = \frac{\sigma'(u)}{\sqrt{3}\alpha^{1/4}} + \frac{\sigma'''(u)}{6\sigma'} \tag{4.32}$$

up to arbitrary translations of the independent variable. Conversely, $\sigma(u)$ can be obtained in terms of $W(u)$ as

$$\sigma(u) = \frac{[\sqrt{3}\alpha^{1/4}W'(u) + (W/\beta)(1+\alpha^{3/2}) + (a^{3/2}-1)/(6\alpha^{1/2})]}{[2W - (\alpha^{3/2}+1)^2/(6\beta^2\alpha^{1/2})]}, \tag{4.33}$$

which constitutes an explicit solution for the unknown $\sigma(u)$ as a function of the integration constant α; as already remarked, the function $W(u)$ does not depend on α, and thus *the solution's dependence on the integration constant is purely algebraic*; we shall return to that point in the next section.

It is worth noting that the formula (4.33) can be considerably simplified through the introduction of two new constants λ and μ:

$$\lambda = \frac{(\alpha^{3/2}+1)^2}{12\beta^2\alpha^{1/2}}; \quad \mu = \lambda^{3/2} + \frac{\alpha^{3/2}-1}{12\sqrt{3}\alpha^{3/4}}, \tag{4.34}$$

which are related by

$$4\mu^2 = 4\lambda^3 - \tfrac{1}{3}m\lambda - \tfrac{1}{27}. \tag{4.35}$$

In terms of these, one may introduce the 'spectral-function' $\chi(u)$ associated with the Weierstrass function, satisfying the pair of equations:

$$\frac{2\chi'}{\chi} = \frac{W'+2\mu}{W-\lambda}, \quad \frac{\chi''}{\chi} = 2W+\lambda \tag{4.36a, b}$$

(the latter having the form of a one-dimensional Schrödinger equation for the wave-function χ); the formula (4.33) then takes the simple form

$$\frac{\sigma(u)}{\sqrt{3}\alpha^{1/4}} = \frac{\chi'}{\chi} + \lambda^{1/2}. \tag{4.37}$$

When $m = 1$ (i.e. $\hat{E} = \frac{9}{2}$), the elliptic solutions degenerate to trigonometric ones; this interesting special case is presented in Appendix D.

4.4. *The general solution ($\hat{E}$ and I_2 both arbitrary)*

We have seen in §4.2 that the first-order o.d.e. (4.8) to which the problem has been reduced, is expected to have the integrating factor $\partial(m, I_2)/\partial(Y, Z)$ (Jacobian taken at U and V constant). The partial derivatives involved being

$$\frac{\partial m}{\partial Y} = \frac{Z}{2}\left(\frac{U}{V}\right)^{1/2}(\beta U - \alpha), \quad \frac{\partial m}{\partial Z} = \frac{Z}{2}(\beta V + 1); \tag{4.38a}$$

$$\frac{1}{V^{1/2}}\frac{\partial I_2}{\partial Y} = \frac{3Z^2}{4V}(\alpha U^2 - \beta) + \left(\frac{(U^3-1)}{U}\right), \quad \frac{1}{U^{1/2}}\frac{\partial I_2}{\partial Z} = \frac{3\alpha Z^2}{4V}(V^2+\beta) + \frac{(V^3-1)}{V}, \tag{4.38b}$$

one obtains the following result for the Jacobian:

$$\frac{2V^{1/2}}{Z}\frac{\partial(m, I_2)}{\partial(Y, Z)} = \Psi, \tag{4.39}$$

where, by definition,

$$\Psi \equiv (N_\alpha - U^2 N_\beta) + 3I_2 \frac{U^{1/2}}{Z}(\beta U - \alpha). \tag{4.40}$$

Thus equation (4.8) admits the integrating factor Ψ, and may be written in the exact differential form (see (4.9))

$$\mathrm{d}\Phi = \frac{V\mathrm{d}V - \alpha U \mathrm{d}U}{\Psi}, \tag{4.41}$$

generalizing the result (4.22) of the preceding subsection.

The above equation (4.41), $\mathrm{d}\Phi = 0$, is an o.d.e. for the unknown function $V(U)$, α being an implicit function of U and V determined by the equations (4.10), (4.15), (4.20); to verify that Ψ is really an integrating factor, we need the partial derivatives of α, which are found to have the form

$$\left.\begin{aligned}
\frac{\partial \alpha}{\partial U} &= \frac{-2\alpha}{U} + \frac{\alpha}{2\Psi}\left(\frac{4N_\alpha}{U} - \frac{9\alpha N}{D}\right) + \frac{3N(U^2 - mV)}{2\Psi(3mUV - \delta)},\\
\frac{\partial \alpha}{\partial V} &= \frac{+2\alpha}{V} + \frac{6}{\Psi}\left(\frac{(m\alpha\beta - 1)}{V} - \frac{3N}{4D}\right) + \frac{3N(V^2 - mU)}{2\Psi(3mUV - \delta)}.
\end{aligned}\right\} \tag{4.42}$$

Another quantity of interest is the total derivative:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}u} \equiv \frac{U'(u)}{V}\left[V\frac{\partial \alpha}{\partial U} + U\alpha\frac{\partial \alpha}{\partial V}\right].$$

If we note that an alternative expression of Ψ reads

$$\tfrac{1}{3}\Psi \equiv V(mV - U^2) + \alpha U(mU - V^2) + \alpha(U + \alpha V)(\delta - 3mUV)/D \tag{4.43}$$

we obtain for $\mathrm{d}\alpha/\mathrm{d}u$ the simple result

$$\frac{V}{U'(u)}\frac{\mathrm{d}\alpha}{\mathrm{d}u} = V\frac{\partial \alpha}{\partial U} + U\alpha\frac{\partial \alpha}{\partial V} = \frac{N/2}{(\delta - 3mUV)} \tag{4.44}$$

or, equivalently,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}u} = \frac{-2I_2}{3UZ^2}.$$

It is easily seen that the condition that Ψ must satisfy in order to be the integrating factor in equation (4.41) is

$$\frac{\mathrm{d}\ln\Psi}{\mathrm{d}u} = U^{3/2}Z\frac{\partial\alpha}{\partial V}; \tag{4.45}$$

direct computation of $\mathrm{d}\Psi/\mathrm{d}u$ shows that the equality does hold, as it should.

Unlike the case where $I_2 = 0$, it does not seem possible to perform the required quadrature (4.41) in closed form, α being no longer constant; the functions $U(u)$, $V(u)$ cannot be obtained explicitly either. We remark however, that the system considered, having the Painlevé property, is expected to have a Bäcklund transformation; the latter, although of non-algebraic form in general, is expected to become algebraic in the case of o.d.e.s; the existence of such a transformation would naturally explain the algebraic relation between solutions found in the case $I_2 = 0$ (equation (4.33)) and the occurrence of the over-determined spectral function χ (equations (4.36), (4.37)); it would similarly *entail an algebraic relation between the solutions of (4.41) corresponding to different values of the integration constant* Φ. We expect that, in its infinitesimal limit, the Bäcklund transformation should be a combination of (infinitesimal) translations of u and of the symmetry generator associated with infinitesimal variations of Φ; its complete determination, in finite form, is currently in progress.

5. Discussion

The discussion will be restricted for simplicity to the cases where the first integral I_2 vanishes:

$$I_2 = 0;$$

such solutions involve the two integration constants, α, β, a third constant m which is related to them by

$$N(\alpha,\beta) \equiv \alpha^3+\beta^3+3m\alpha\beta-1 = 0, \tag{5.1}$$

and they have been shown to be describable by elliptic functions.

5.1. *The physical domain of variation of the integration constants*

First we observe that the constants may not be chosen fully arbitrarily: there are unphysical regions in the (α,β)-plane which must be excluded, as we now show. To start with, equation (4.18) manifestly excludes the quadrant $\{\alpha < 0,\ \beta < 0\}$. Another important constraint is that the variables (U, V) must always be *positive*, since H and K must be real; let us then consider the potential energy term, δ/UV, in the integral of energy (equation (4.15)): we have the identity

$$\delta-3UV \equiv (U^3+V^3+1)-3UV \equiv (U+V+1)[U^2-UV+V^2-U-V+1], \tag{5.2}$$

where the second factor is a quadratic form that always remains positive (vanishing only when $U = V = 1$), and the first factor is positive too, since U and V take only positive values; the potential energy term is therefore bounded from below:

$$\frac{\delta}{UV} \geqslant 3. \tag{5.3}$$

The kinetic energy term being of course positive, the energy integral (4.15) thus provides a constraint on $m \equiv 2\hat{E}/9$:

$$m \geqslant 1.$$

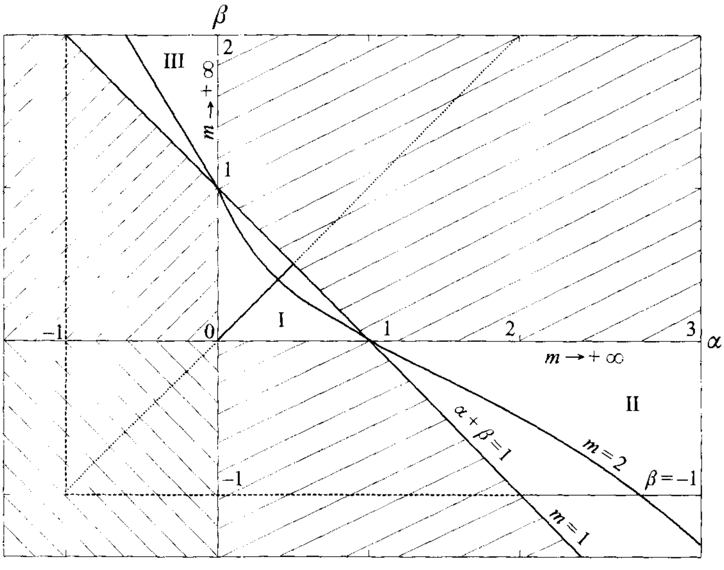

FIGURE 1. There are three physically meaningful regions I, II and III in the plane of the integration constants α and β ($I_2 = 0$); the curves $m(\alpha, \beta) =$ constant fill these three regions when m varies from $+1$ to $+\infty$. The three regions are interchanged by permutation of the ellipsoid's axes a, b, c and therefore represent identical flows. The lines $\alpha = -1$ and $\beta = -1$ are the images of the symmetry axis $\alpha = \beta$, by permutation; all three lines intersect at ($\alpha = -1$, $\beta = -1$), which is an isolated real point on the curve $m(\alpha, \beta) = 1$.

That excludes three regions in the (α, β)-plane:

a region in the quadrant $(\alpha > 0, \beta > 0)$, where $\alpha+\beta > 1$,

a region in the quadrant $(\alpha > 0, \beta < 0)$, where $\alpha+\beta < 1$,

a region in the quadrant $(\alpha < 0, \beta > 0)$, where $\alpha+\beta < 1$.

This leaves only three physically accessible regions, denoted I, II and III (see figure 1); furthermore, these three regions in fact represent the same physical flows, being related by the permutation group of the axes a, b, c discussed in Appendix B: the permutation (S^*), which exchanges a and b, also exchanges α and β, and accordingly maps region II into region III; and the permutation $(\tilde{S})$, which exchanges b and c, operates on α, β according to

$$\tilde{\alpha} = 1/\alpha, \quad \tilde{\beta} = -\beta/\alpha, \tag{5.4}$$

thus mapping region I into region II.

That symmetry property enables us to exclude without loss of generality region III (where $\alpha < 0$) from our consideration; the constant $\alpha^{1/2}$ which occurs in the parametrization (4.24) of the trajectories is then *real*.

The curves $m(\alpha, \beta) =$ constant play a fundamental role; their essential properties are described below:

The curve $m(\alpha, \beta) = 1$ is the straight line $\alpha+\beta = 1$; this is the limit line where the elliptic function $W(u)$ becomes *trigonometric*.

The curves $m(\alpha, \beta) = C$, when the constant C increases without limit, get closer and closer to the axes $\alpha = 0$ and $\beta = 0$; in the limit $m \to \infty$ the elliptic function $W(u)$ becomes *lemniscatic* (its two periods have the same modulus).

The points ($\alpha = 0$, $\beta = 1$) and ($\alpha = 1$, $\beta = 0$) represent the *axisymmetric solutions*.

The curves $m(\alpha,\beta) = C$ have an asymptote:

$\alpha+\beta = C$;

the point at infinity represents the axisymmetric solutions again (it is the image of the point ($\alpha = 0$, $\beta = 1$) under the permutation $(\tilde{S})$).

The curves $m(\alpha,\beta) = C$ are symmetrical with respect to the axis $\alpha = \beta$; the points on the axis – in the physically allowed range $0 < \alpha, \beta < \frac{1}{2}$ – represent solutions with a special symmetry property, discussed below.

Each trajectory in the (U, V)-plane presents several stationary points S_i ($i = 0, 1$, etc.) where $U'(u) = V'(u) = 0$. At such points the velocity variables X, Y, Z must vanish, hence $Y/Z \to Y'_0/Z'_0$, where the values of Y'_0 and Z'_0 are obtainable from equation (4.4); the definitions (4.13) of α and (4.18) of β then give the result

$$\alpha = \frac{U_0}{V_0}\left(\frac{V_0^3-1}{U_0^3-1}\right), \quad \beta = \frac{(U_0^3-V_0^3)}{V_0(U_0^3-1)}. \tag{5.5}$$

The corresponding value of m follows from (4.11), but it may be found much more simply using (4.15):

$$m = \frac{U_0^3+V_0^3+1}{3U_0 V_0}. \tag{5.6}$$

Conversely one may need to determine the stationary points (U_i, V_i), given the values of α and β; they are given by the parametric representation (4.24), where σ is any one of the four roots $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ of the quartic $P_4(\sigma)$; there are thus *four stationary points*† – but *two of them only are physically meaningful*, as shown by the discussion below.

The four values of U_i ($i = 0, 1, 2, 3$) are the roots of a fourth-degree polynomial $R_4(U)$:

$$R_4(U) \equiv \beta U^4 + 2\alpha U^3 - 3mU^2 + 2\beta U + \alpha = 0, \tag{5.7}$$

and the corresponding values of V_i are explicitly given by

$$V_i = U_i/(\alpha+\beta U_i). \tag{5.8}$$

Let us first consider the case where (α,β) is in region I – hence $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha+\beta < 1$. We obtain

$$\left.\begin{aligned} &R_4(+\infty) = +\infty,\\ &R_4(1) = 3(\alpha+\beta-m) < 0 \quad (\text{since}\, m > 1 \quad \text{and} \quad \alpha+\beta < 1),\\ &R_4(0) = \alpha > 0,\\ &R_4(-1) = -(\alpha+\beta+3m) < 0,\\ &R_4(-\infty) = +\infty, \end{aligned}\right\} \tag{5.9}$$

so there exist two positive roots, U_0 and U_1, and two negative roots, U_2 and U_3; the latter are of course unphysical.

Choosing now the point (α,β) in region II instead of region I merely exchanges the roles of U and V, and we conclude that, in region II, the four values V_i of V at the stationary points must be given by a quartic equation with four real roots, two positive (V_0 and V_1) and two negative (V_2 and V_3). Thus in all cases *there are two and only two stationary points* (S_0 *and* S_1) *which are physically meaningful, and the evolution must*

† These four points delineate a period parallelogram of the associated elliptic functions.

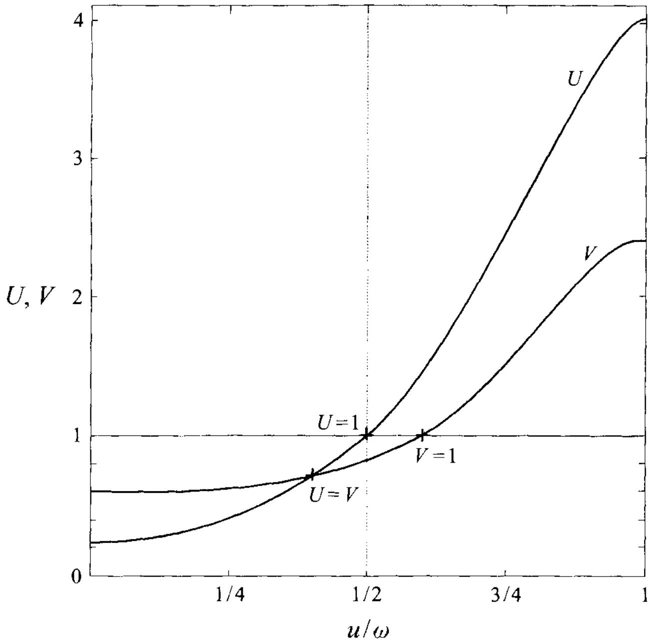

FIGURE 2. The variation of the ellipsoid's shape ($U \equiv (b/a)^{2/3}$, $V \equiv (c/a)^{2/3}$) during a half-period ω of the independent variable u (= thermasy), assuming the values $\alpha = 1/3$, $\beta = 1/3$, $I_2 = 0$ for the integration constants. If the expansion starts from rest at $u = 0$, it will terminate ($t \to \infty$) at a point u_∞ between a quarter- and half-period: $\frac{1}{2}\omega < u_\infty < \omega$.

consist of an oscillation between these two points (figure 2). Whether a full oscillation effectively takes place, or more, or less, is a distinct problem that will be addressed later.

Let us examine more precisely the form of this oscillation, in the two limits of the range of values of m. First, in the trigonometric limit ($m \to 1$, $\alpha+\beta \to 1$), the polynomial $R_4(U)$ has a double root:

$$\begin{aligned} R_4(U) &\equiv (1-\alpha)\,U^4 + 2\alpha U^3 - 3U^2 + 2U(1-\alpha) + \alpha \\ &\equiv (U-1)^2\,[(1-\alpha)\,U^2 + 2U + \alpha]. \end{aligned} \tag{5.10}$$

The two positive roots are thus $U_0 = 1 = U_1$; consequently the motion, as $m \to 1$, consists of an oscillation of infinitesimal amplitude ϵ between the two roots:

$1-\epsilon \leqslant U \leqslant 1+\epsilon$. The corresponding roots V_0, V_1 also tend towards unity: the expansion of the cloud proceeds with nearly spherical symmetry.

In the lemniscatic limit, we have: $\beta \to 0$, $\alpha > 0$ and $m \sim (1-\alpha^3)/(3\alpha\beta) \to +\infty$; for definiteness, we shall choose the point (α, β) in region I, so we have

$\beta \to 0^+, 0 < \alpha < 1$. The polynomial equation $R_4(U) = 0$ degenerates to $U^2 = 0$, which has the double root $U = 0$, the two remaining roots being removed at infinity. More precisely, the four roots are asymptotically arranged in the order $\{-\infty, 0^-, 0^+, +\infty\}$:

the motion therefore consists of an oscillation between a very small value U_0 and a very large one U_1. Asymptotically, we obtain

$$U_0 \sim \alpha\left(\frac{\beta}{1-\alpha^3}\right)^{1/2}; \quad V_0 \sim \frac{U_0}{\alpha}, \qquad U_1 \sim \frac{1-\alpha^{3/2}}{\beta\alpha^{1/2}}; \quad V_1 \sim U_1\,\alpha^{1/2}. \tag{5.11}$$

Thus, starting from an extremely flat disk configuration (S_1), the motion proceeds towards a very elongated cigar-shaped one (S_0), and vice versa; again, the question of what fraction of an oscillation effectively takes place is not immediately answerable without calculation.

To summarize, the oscillation has an amplitude that decreases progressively as $m \to 1$ (where the expansion becomes purely spherical), and increases without limit as $m \to \infty$.

Having discussed the effect of varying m, we now turn to a consideration of the effect of varying the remaining parameter α, m being kept fixed. There is no loss of generality in assuming the representative point (α, β) to lie in region I; further, owing to the symmetry about the diagonal axis $\alpha = \beta$, we may even select the upper-half of region I, where $\beta > \alpha$. In this region a curve $m(\alpha, \beta) = \text{constant}$ is bounded by the point $(\alpha = 0, \beta = 1)$ and by another point on the main diagonal where $\alpha = \beta$. The first point represents the limiting case of a purely axisymmetric expansion, with $V \equiv 1$, i.e. $a \equiv c$; the qualitative behaviour of such flows is known and will be briefly summarized below (§5.2). At the second point, where the integration constants α and β have equal values, and the coordinates (U_0, V_0) at a stationary point are accordingly related by

$$V_0^3 = U_0(U_0^2+1)/(U_0+1), \tag{5.12}$$

the trajectories in the (U, V)-plane are invariant under the permutation (S^*). In particular, the stationary point S_1 is the image of S_0 under (S^*) – that is to say, we must have

$$U_1 = 1/U_0, \quad V_1 = V_0/U_0. \tag{5.13}$$

The relation between S_0 and S_1 being a mere permutation of the axes a, b, c, we see that *the shape of the ellipsoidal cloud is the same in the initial state* S_0 *as in the opposite limit* S_1 of the oscillation. More generally, letting the elliptic independent variable u take values $u_0 = 0$ at S_0, and the real half-period value $u_1 = \omega$ at S_1, the gas cloud assumes the same shape at values of u symmetrically disposed with respect to the quarter-period value, $u = \omega/2$. It must be noted that, although its shape remains the same, *the ellipsoid's orientation differs by a* 90° *rotation*, since the axes a and b have been permuted.

Let us finally mention that the diagonal $\alpha = \beta$ has images (by the permutation group) in regions II and III, which are the lines $\beta = -1$ and $\alpha = -1$, respectively.

5.2. *The end point of the expansion*

The mathematical theory developed in §4 gives us the evolution of the shape of an ellipsoidal cloud (i.e. of U and V) as a function of the thermasy u. The scale R, and the correspondence between u and physical time t, are obtained through the following steps. First we compute the modified time variable t^*, given by

$$t^* = \int \frac{UV}{\delta} \mathrm{d}u + \text{constant}, \tag{5.14}$$

choose an arbitrary value of the energy constant E, and obtain

$$R = \left(\frac{\hat{E}}{E}\right)^{1/2} \frac{1}{\cos((2\hat{E})^{1/2} t^*)} \tag{5.15}$$

from integration of $\mathrm{d}R/\mathrm{d}t^* = R[2(ER^2 - \hat{E})]^{1/2}$ (see (3.13), (3.23)); finally, the physical time t is found:

$$t = \frac{1}{E}\left(\frac{ER^2 - \hat{E}}{2}\right)^{1/2} + \text{constant} \tag{5.16}$$

(that is just the solution of the trinomial equation (3.10)).

It is essential to remark that, although the evolution may proceed an infinitely long time t, the corresponding range of values of the elliptic phase u remains finite: $u \to u_\infty$ as $t \to \infty$; therefore, only a finite number of oscillations between the limit points S_0 and S_1 may occur; and, as it turns out, *the oscillation proceeds for less than a half-period*, at least in several important cases.

We have seen that the moment of inertia R^2 is related to the modified time variable t^* (the canonical time appropriate to the reduced motion on the surface of the 2-sphere) as

$$R^2 = \frac{\hat{E}/E}{\cos^2(3t^* m^{1/2} + \psi_0)}, \tag{5.17}$$

where ψ_0 is an arbitrary constant phase.

Considering for simplicity the case of an expansion starting from rest at the stationary point S_0 at time $t^* = 0$, we must set $\psi_0 = 0$, since $\dot{R}$ then initially vanishes (recall that $\dot{R}$ does not necessarily vanish, at a stationary point S_i: it is only $\dot{U}$ and $\dot{V}$ that have to be zero there); the end point of the expansion therefore occurs at

$$t^* = t^*_\infty = \frac{\pi}{6m^{1/2}}. \tag{5.18}$$

The corresponding value u_∞ of the elliptic phase u (starting from $u = 0$ at point S_0, and hence $u = \omega$ at point S_1) is given by (see equation (5.14)):

$$t^*_\infty = \int_{u=0}^{u_\infty} \frac{UV}{1 + U^3 + V^3} \mathrm{d}u = \frac{\pi}{6m^{1/2}}. \tag{5.19}$$

Finding the endpoint of the expansion thus involves calculating the above elliptic integral; complicated as it may seem, it is nevertheless integrable in terms of the σ-functions of the elliptic theory.

To perform the required integration, one should first re-express the integrand in terms of the Weierstrass function $W(u)$: this is achieved simply by means of the parametric representation (4.24), where the parameter σ itself is known to be a homographic function of $W(u)$; as a result, U must have the form

$$U = P_2(W)/Q_2(W), \tag{5.20}$$

where P_2 and Q_2 are second-degree polynomials; and the integrand UV/δ must have the general form

$$UV/\delta = P_6(W)/Q_6(W), \tag{5.21}$$

where P_6, Q_6 are polynomials of the sixth-degree. The rational fraction P_6/Q_6 should then be decomposed into simple elements, and the result will qualitatively depend upon whether the denominator Q_6 has multiple roots or not; this is then a point worth investigating.

In fact, it is simpler to write down the equation $Q_6(W) = 0$ in terms of the variable σ rather than W; let $R_6(\sigma) = 0$ be the resulting (sixth-degree) polynomial equation in σ: since σ is a homographic function of W, the questions of whether Q_6 and R_6 have multiple roots are equivalent. We obtain

$$R_6(\sigma) \equiv (\alpha^{3/2} - 1)\sigma^6 + 3\beta\sigma^4(\alpha^{3/2} + 1) + 8\alpha^{3/2}\sigma^3 + 3\beta^2\sigma^2(\alpha^{3/2} - 1) + \beta^3(\alpha^{3/2} + 1) \tag{5.22}$$

and its discriminant is the product of factors:

$$(\alpha^3-1)(\beta^3-1)(\alpha^3+\beta^3)(\alpha^3+\beta^3-1). \tag{5.23}$$

It is easily seen that, the point (α, β) being constrained to lie in one of the physically allowed regions, this discriminant vanishes only at three points: $(\alpha = 0, \beta = 1)$, $(\alpha = 1, \beta = 0)$ and $(\alpha = 0 = \beta)$; we conclude that, with the possible exception of these three points, the calculation of t^* is fully reducible to integrals of the type

$$\int \frac{\mathrm{d}u}{W(u)-a},$$

where a is a constant.

In the axisymmetric case, with, e.g. $U \equiv 1$, the variable V itself is the Weierstrass function, except for a constant normalization factor:

$$V \equiv -6W(u), \tag{5.24}$$

and the integral giving t^* becomes simpler:

$$t^* = \int \frac{V\,\mathrm{d}u}{V^3+2}. \tag{5.25}$$

The axisymmetric case was also considered in Dyson's work (1968, p. 100), where it is stated that the evolution changes a cigar-shaped cloud into a disk-shaped one, and vice versa; that means that u_∞ must lie somewhere between a quarter-period and a half-period:

$$\tfrac{1}{2}\omega < u_\infty < \omega. \tag{5.26}$$

The only case where a definite answer can be obtained in a simple way is the trigonometric limit $(m \to 1)$, where $t^* \equiv u/3$, and hence $u_\infty = \pi/2$.

The half-period ω is generally given by (see Appendix C):

$$\omega = \frac{K(\nu)}{(W_1 - W_3)^{1/2}} \tag{5.27}$$

where $W_3 < W_2 < W_1$ are the three real zeros of $W'(u)$, in that order, $\nu \equiv (W_2 - W_3)/(W_1 - W_3)$ is a universal (and algebraic) function of m only, and $K(\nu)$ is the well-known complete elliptic integral of the first kind, which takes the value $K(0) = \pi/2$ at $\nu = 0$.

When $m = 1$, we obtain: $\nu = 0$, $W_1 = 1/3$, $W_2 = W_3 = -1/6$, and the half-period $\omega = \pi/\sqrt{2}$; hence the oscillation, starting from rest at a stationary point, terminates at elliptic phase:

$$u_\infty = \omega/\sqrt{2}, \tag{5.28}$$

again a value intermediate between a half- and quarter-period.

5.3. *The evolution of oblateness in the course of expansion*

The trajectories in the (U, V)-plane present *the remarkable property of passing three times* (during a half-period) *through an axisymmetric configuration*: first a prolate spheroid, then an oblate, then again a prolate configuration. This general behaviour may be illustrated with the following two typical examples chosen in the lower-half of region I (where $\alpha \geqslant \beta$). The first example corresponds to a point on the main diagonal: $\alpha = \beta = 1/3$, $m = 25/9$; while the second, with parameters $\alpha = 0.9$, $\beta = 0.05$, $m = 2.006\ldots$, lies close to the axisymmetric limit at $(\alpha = 1, \beta = 0)$ where the functions $U(u)$ and $V(u)$ become identical.

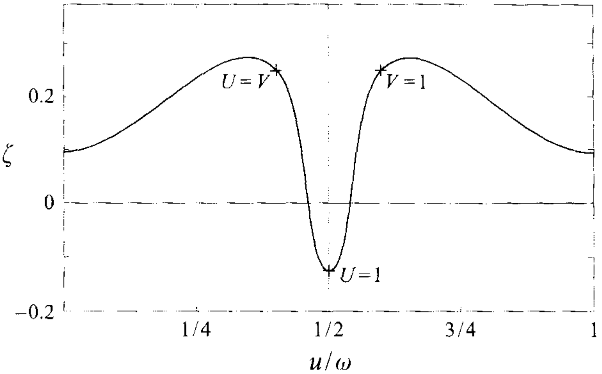

FIGURE 3. Evolution of the measure of oblateness ζ (defined in Appendix E) for the case $\alpha = 1/3 = \beta$, $I_2 = 0$, during a half-period $0 < u < \omega$. (For prolate (resp. oblate) spheroids, $\zeta > 0$ (< 0)). Cases $\alpha = \beta$ are characterized by a symmetrical evolution with respect to the quarter-period axis. The three crosses mark the passage through an axisymmetric configuration.

We have plotted in figures 3 and 4 the evolution during a half-period of the shape parameter ζ introduced in Appendix E, which is a measure of the oblateness of the ellipsoidal cloud (ζ varies from -1 for a flat disk to zero for a sphere and $+2$ for an extreme prolate spheroid). The most striking feature of figure 3 is its exact symmetry about the axis $u = \omega/2$ (as predicted in §5.1); in that case the 'half-period' becomes a full period. The generally decreasing profile of figure 4 prefigures the exactly monotonic result appropriate to the axisymmetric flows. Two out of the three axisymmetric configurations, at $U = 1$ and $V = 1$ respectively, are nearly spherical; whereas the other, at $U = V = 1/\sqrt{2}$ in our example, tends in general as $\beta \rightarrow 0$ to the spheroidal limit $U = V = 1/m^{1/2}$. Choosing values of (α, β) less close to the axisymmetric limit would of course result in a diagram for $\zeta(u)$ intermediate between those of figures 3 and 4.

6. Conclusion

We have here considered Dyson's model of an adiabatically expanding ellipsoidal gas cloud, and have shown that, under certain restricting assumptions, it turns out to be *a completely integrable model.* The crucial assumption leading to integrability appears to concern the equation of state of the fluid, which we have taken to be the monatomic ideal gas, characterized by an adiabatic index $\gamma = (N+2)/N$ in a space of dimension N – hence $\gamma = 5/3$ (although the values $\gamma = 2$ and 3 may also be physically relevant, in the more special cases of planar motion and of one-dimensional motion). We have in addition neglected for simplicity the effects of rotation, so that the integrable model presented here represents a deformable expanding cloud of tri-axial ellipsoidal shape, the principal axes maintaining a fixed orientation in space.

The clue to uncovering the integrability property of this model was the realization that some of the physical variables involved were uniform functions of a well-chosen independent variable – not the time t, but the thermasy: $u = \int T\,\mathrm{d}t$, which is one of the potentials occurring in the Clebsch transformation of the velocity field. In other words, the evolution of the system was governed by differential equations whose general solution was free of (movable) singularities other than poles: such systems are said to possess the Painlevé property (Ince 1956) and have been conjectured to be completely integrable (Ablowitz & Segur 1977). Unfortunately, for ordinary differential systems

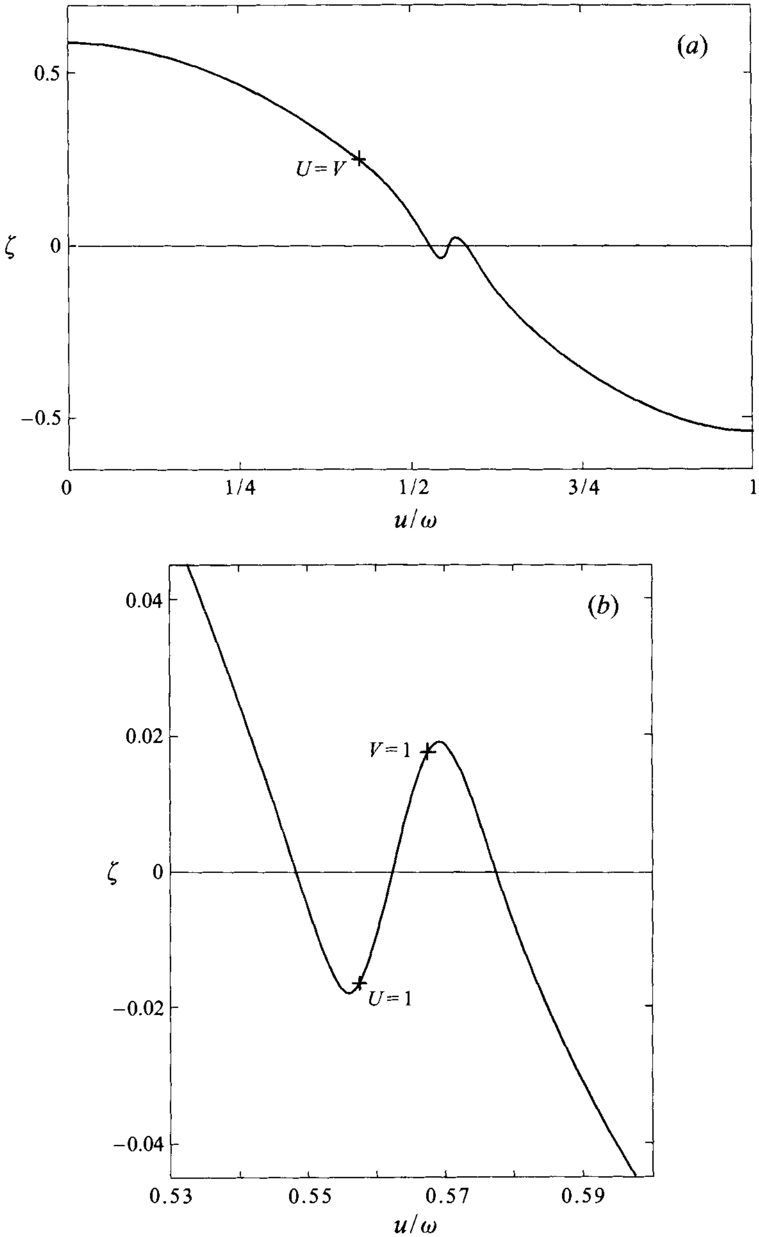

FIGURE 4. (*a*) Same as figure 3, for the case $\alpha = 0.9$, $\beta = 0.05$ ($I_2 = 0$), close to the spheroidal limit $\beta \rightarrow 0$. Note the nearly monotonic variation of ζ, characteristic of axisymmetric ellipsoids. (*b*) A magnified view of the central kink in (*a*); two of the three axisymmetric configurations (marked by crosses) are nearly spherical and coalesce in this small central region, as $\beta \rightarrow 0$.

(which is the case here), the Painlevé conjecture provides no systematic method to perform in effect the integration. We have nevertheless been able to construct the missing first integrals and to complete the integration by reducing it to quadratures.

The solutions obtained have an essentially different character according to whether the second integral of the motion (denoted I_2) vanishes or not. We remark that, for motions starting from a state of rest, the second integral I_2 does vanish, being of odd parity in the velocities (see (4.5)); such motions are describable by elliptic functions. The elliptic case describes even more general flows: it is sufficient, in order that $I_2 = 0$, that the initial condition be a state of homologous expansion, i.e. that all three principal axes be initially expanding in the same proportion.

When $I_2 \neq 0$ on the other hand, no such simple result obtains; we expect that, given one solution arbitrarily, there should exist a one-parameter family of new solutions algebraically related to it, as a result of the existence of a Bäcklund transformation (see §4.4); this, however, remains to be confirmed.

Appendix A. The Painlevé property

The question of the integrability of a differential system has long been known to be connected with the system being endowed (or not) with the Painlevé property (Ince 1956). That is the property of a system whose generic solution admits only pole singularities in the complex plane of the independent variable (u, say), with the possible exception of a certain number of singularities of another type (branch points, etc...) at fixed locations in the complex plane. It has been conjectured (Ablowitz & Segur 1977) that any system passing the Painlevé test must be completely integrable, in a certain sense. One of the main practical difficulties in establishing the Painlevé property lies in the appropriate choice of the independent variable; it is relatively easy to show that, in the case of an axisymmetric ellipsoidal gas cloud ($H \equiv K$), the correct choice is to identify u with the *thermasy*, and that *the function* $U \equiv H^{2/3}$ *then possesses the Painlevé property*, being an elliptic function of u. We now extend this result to the general case of a tri-axial ellipsoidal cloud.

Let us start with the formulation (4.2) of the system: a search for the singularities of the solutions $U(u)$ to (4.2) brings to light the following two types:

Case (i): $U \sim a_0/u$, where a_0 is a non-zero constant, and $u \to 0$ (there is no loss of generality in assuming the singularity located at $u = 0$, in view of the invariance of the system under arbitrary translations of u). The function V has then a simple pole singularity as well:

$$V \sim b_0/u, \quad \text{where} \quad a_0 b_0 = -3/4.$$

Case (ii): $U \sim -6/u^2$, and then $V \to 1$. (There exists, of course, another singularity type symmetrical to Case (ii), in which $V \sim -6/u^2$ and $U \to 1$. The reason we do not include it in the list of singularity types of the function $U(u)$, is that the latter then remains perfectly regular.)

Let us then proceed with the Painlevé test, starting with the singularity type (i); the test requires pursuing the expansion until all free parameters (= integration constants) have been found; the result is as follows:

$$U = a_0/u + a_1 + a_2 u + ku^2 + \dots, \quad V = b_0/u + b_1 + b_2 u + lu^2 + \dots, \tag{A 1}$$

where a_0, a_1, a_2 may be viewed as the three integration constants (the fourth being provided by the arbitrary translations of u), and b_0, b_1, b_2 are related to them by the symmetrical formulae:

$$a_0 b_0 = -3/4, \quad a_0 b_1 = a_1 b_0, \quad a_0 b_2 - 2a_1 b_1 + a_2 b_0 = 0. \tag{A 2}$$

The u^2 terms, with coefficients k, l, are shown here only to indicate the absence of logarithmic terms $u \ln u$ that might have occurred; the values of the constants k, l are functions of the integration constants, and need not be explicited. The essential point is the presence of the full complement of four integration constants in the expansion

(A 1), meaning that it represents the behaviour of the generic solution to (4.2). Thus, (4.2) does pass the Painlevé test, as far as the present singularity branch – referred to as branch (i) hereafter – is concerned.

Turning to Case (ii): when the leading term in the singular expansion is the double pole: $U \sim -6/u^2$, we obtain the following result:

$$U = \frac{-6}{u^2} - 3\lambda u + \mu u^2 + k u^4 + \dots, \quad V = 1 + \lambda u^3 + l u^6 + \dots, \tag{A 3}$$

where the constants λ and μ are arbitrary. This expansion thus involves only two arbitrary parameters (λ, μ) instead of the three required for the expansion to be generic. In fact, linearization in the form

$$U = \frac{-6}{u^2}(1 + \epsilon u^n) \quad (\epsilon \to 0, u \to 0)$$

yields the values of the exponent n (called *resonances*) where the arbitrary parameters must occur; they are, in this case,

$$n = -4, \ -1, 3, 4.$$

The value $n = -1$ corresponds to the arbitrary translations of the singular point; the value $n = -4$ however is incompatible with the leading term being $-6/u^2$, and that explains why one integration constant is missing. Thus the expansion of this branch (called branch (ii) in what follows) is *non-generic*. The point however, is that the branch is nevertheless free of logarithmic terms, and of any singularities other than poles, and thus it does not spoil an eventual Painlevé property of the system.

In conclusion, the Painlevé test applied to both singularity branches (i) (generic), and (ii) (non-generic) indicates that *the system* (4.2) *does possess the Painlevé property* (with the thermasy u as the independent variable), and is therefore completely integrable, if the Painlevé conjecture is indeed valid. The Painlevé conjecture unfortunately does not specify how to proceed with the integration, but it does encourage us to look for further constants of the motion, which should be present if the system is integrable at all.

Appendix B. The permutation group of a, b, c

We must say a word about the presence of discrete symmetries of our system (4.2), which reflect its invariance under the group of permutation of the three principal axes a, b, c.

Let us recall the definition of the unknown functions $U(u)$ and $V(u)$, in terms of ratios of principal axes:

$$U^{3/2} \equiv H \equiv b/a; \quad V^{3/2} \equiv K \equiv c/a. \tag{B 1}$$

Exchanging the roles of a and b, one obtains a first (reciprocal) symmetry of the equations, denoted (S^*), which has the following action on variables:

$$U \to U^* = 1/U, \quad V \to V^* = V/U \tag{B 2}$$

without affecting the independent variable u. On the other hand, if we interchange the roles of b and c, the result is a symmetry $(\tilde{S})$ whose action is simply given by

$$\tilde{U} = V, \quad \tilde{V} = U. \tag{B 3}$$

We observe that the first integral $\hat{E}$ is invariant under the action of both (S^*) and $(\tilde{S})$; in fact, the kinetic and the potential terms in $\hat{E}$ are separately invariant.

It is possible to rewrite the system (4.2) in manifestly invariant form, in the following way. In terms of the Cartesian position vector $\boldsymbol{x}$ introduced in §3.3 ($x \equiv a/R$, $y \equiv b/R$, $z \equiv c/R$, $\boldsymbol{x}^2 = 1$), the equation of motion (4.2*a*) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{(xy'-yx')}{(xyz)^{2/3}}\right] = \frac{x}{y} - \frac{y}{x}.$$

Introduce the 3-vector $\boldsymbol{X}$, with components X, Y, Z, defined by

$$\tfrac{3}{2}\boldsymbol{X} \equiv \frac{\boldsymbol{x} \wedge \boldsymbol{x}'(u)}{(xyz)^{2/3}} \tag{B 4}$$

($\boldsymbol{X}$ is thus proportional to the (non-conserved) angular momentum vector of the point-mass moving on the unit sphere); equation (4.2*a*) becomes

$$\frac{3}{2}\frac{\mathrm{d}Z}{\mathrm{d}u} = \frac{x^2-y^2}{xy} \tag{B 5}$$

and equation (4.2*b*)

$$\frac{3}{2}\frac{\mathrm{d}Y}{\mathrm{d}u} = \frac{z^2-x^2}{zx}. \tag{B 6}$$

To make the system manifestly invariant under permutations of x, y, z, we need only complete the above equations (B 5), (B 6) by a third equation:

$$\frac{3}{2}\frac{\mathrm{d}X}{\mathrm{d}u} = \frac{y^2-z^2}{yz}, \tag{B 7}$$

which may be shown to be a consequence of the other two.

It is worth pointing out finally that, as an immediate consequence of the definition (B 4), the three components X, Y, Z are linked by the simple relation

$$\boldsymbol{x} \cdot \boldsymbol{X} = 0. \tag{B 8}$$

Appendix C. Determination of the main parameters of the flow, given the initial ellipsoidal shape

The initial values a_0, b_0, c_0 of the three principal axes are given, and the motion starts from rest (in fact, $\dot{R}$ need not vanish: it will be sufficient to assume here that the *ratios* b/a, c/a are initially stationary; that is our definition of a 'Stationary point'). The initial shape fixes the coordinates U_0, V_0 of the corresponding stationary point S_0:

$$U_0 = (b_0/a_0)^{2/3}, \quad V_0 = (c_0/a_0)^{2/3}. \tag{C 1}$$

The integration constants α, β that determine the trajectory in the (U, V)-plane are then

$$\alpha = \frac{U_0(V_0^3-1)}{V_0(U_0^3-1)}, \quad \beta = \frac{(U_0^3-V_0^3)}{V_0(U_0^3-1)}, \tag{C 2}$$

and the third constant m, which is a function of α and β (see equation (4.11)) is

$$m = \frac{1-\alpha^3-\beta^3}{3\alpha\beta} = \frac{1+U_0^3+V_0^3}{3U_0V_0}. \tag{C 3}$$

The parameter m fixes the relevant Weierstrass function $W(u)$, whose extrema play a fundamental role and are given by the cubic equation

$$4W^3 - \tfrac{1}{3}mW - \tfrac{1}{27} = 0. \tag{C 4}$$

In the physically allowed range of values of $m(1 < m < +\infty)$, the cubic has three real roots W_1, W_2, W_3, arranged in the order:

$$W_3 < -\tfrac{1}{6} < W_2 < 0 < \tfrac{1}{3} < W_1; \tag{C 5}$$

to find the roots, we introduce an angle ϕ:

$$\cos(3\phi_i) = \frac{1}{m^{3/2}} \quad (i = 1, 2, 3) \tag{C 6}$$

(ϕ is only determined modulo $2\pi/3$, hence the index i; its sign is immaterial), and we obtain

$$W_i = \tfrac{1}{3}m^{1/2}\cos(\phi_i). \tag{C 7}$$

The parameter $\nu = (W_2 - W_3)/(W_1 - W_3)$ plays a fundamental role in the elliptic theory; choosing the appropriate determination of ϕ (that which makes $0 \leqslant \nu \leqslant \frac{1}{2}$), we have

$$\nu = \frac{2}{1 + \sqrt{3}\,\mathrm{cotan}(\phi)}. \tag{C 8}$$

The constant m is algebraically related to ν as

$$\frac{m^3}{4} = \frac{(\nu^2 - \nu + 1)^3}{(\nu - 2)^2(2\nu - 1)^2(\nu + 1)^2}. \tag{C 9}$$

The expression (C 7) for the roots W_i may also be written

$$W_1 = k(2 - \nu)/3, \quad W_2 = k(2\nu - 1)/3, \quad W_3 = -k(\nu + 1)/3, \tag{C 10}$$

where
$$k = \frac{1}{2}\left(\frac{m}{\nu^2 - \nu + 1}\right)^{1/2}.$$

As shown in §5.1, there exist four stationary points S_0, S_1, S_2 and S_3 (two of them, S_2 and S_3 say, falling in unphysical regions), where $U'(u)$ and $V'(u)$ simultaneously vanish; the corresponding values U_0, U_1, U_2, U_3 are the roots of a quartic equation: $R_4(U) = 0$. One of its roots (U_0) is already known, and it turns out that the equation may then be solved exactly:†

$$U_i = \frac{6W_i + U_0V_0}{6W_iU_0^2 + V_0} \quad (i = 1, 2, 3) \tag{C 11}$$

(the fourth root, U_0, may also be found from this formula by substituting the value zero for W_i). By symmetry, the values V_i of V read

$$V_i = \frac{6W_i + U_0V_0}{6W_iV_0^2 + U_0}. \tag{C 12}$$

† Its resolution involves finding the value of an elliptic function for *half the value of the argument*.

The values σ_i of the parameter σ on the trajectory are the roots of the quartic equation: $P_4(\sigma) = 0$ (§4.3); they are found in the form

$$\sigma_i = (V_i - U_i \alpha^{1/2}) \quad (i = 0, 1, 2, 3), \tag{C 13}$$

and may also be written

$$\sigma_i = \sigma_0 \frac{6W_i + U_0 V_0 + l_0}{6W_i + U_0 V_0 - l_0} \quad (i = 0, 1, 2, 3, \text{ letting } W_0 = \infty) \tag{C 14}$$

where

$$l_0 = \left(\frac{(U_0^3 - 1)(V_0^3 - 1)}{U_0 V_0}\right)^{1/2} \quad \left(\text{more precisely, } l_0 = \frac{V_0^3 - 1}{V_0 \alpha^{1/2}}\right). \tag{C 15}$$

Moreover, equation (C 14) remains valid, not only at stationary points, but all along the trajectory; we have indeed *the homographic (Moebius) relation between variables σ and W*:

$$\sigma \equiv \sigma_0 \frac{6W + U_0 V_0 + l_0}{6W + U_0 V_0 - l_0}. \tag{C 16}$$

Finally, it would be useful to also have an explicit expression for U and V in terms of the Weierstrass function $W(u)$. We introduce for conciseness the notation:

$$w \equiv 6W + U_0 V_0, \tag{C 17}$$

so the expression (C 16) of σ simplifies to

$$\frac{\sigma}{\sigma_0} \equiv \frac{w + l_0}{w - l_0}. \tag{C 18}$$

From the parametric representation (4.24), we expect that U should have the form

$$U \equiv \frac{P_2(W)}{w^2 - l_0^2}.$$

P_2 being a second-degree polynomial. Under the permutation (S^*) of the axes a, b (Appendix B) U is changed to $1/U$, therefore $P_2(W)$ must be proportional to $(w^{*2} - l_0^{*2})$, where

$$l_0^* = \frac{1 - U_0^3}{U_0^2} \beta^{1/2}, \tag{C 19}$$

and

$$w^* \equiv 6W + V_0/U_0^2; \tag{C 20}$$

in this way we obtain the explicit expression for U in terms of $W(u)$:

$$\frac{U}{U_0} \equiv \frac{w^{*2} - l_0^{*2}}{w^2 - l_0^2}, \tag{C 21}$$

with w, w^* related to $6W$ by the translations (C 17), (C 20).

Application of the permutation $(\tilde{S})$ yields the corresponding result for V:

$$\frac{V}{V_0} \equiv \frac{\bar{w}^2 - \bar{l}_0^2}{w^2 - l_0^2}, \tag{C 22}$$

where

$$\bar{w} \equiv 6W + \frac{U_0}{V_0^2}, \quad \bar{l}_0^2 = -\frac{\beta}{\alpha} \frac{(V_0^3 - 1)^2}{V_0^4}. \tag{C 23}$$

Appendix D. The trigonometric solutions

The solutions presented below are solutions in real numbers of our differential system (4.2) for the unknown functions $U(u)$, $V(u)$ – but the values of U and V are not simultaneously positive: one of the ratios $H \equiv b/a$, $K \equiv c/a$ must accordingly be complex. These solutions may nevertheless gain physical meaning in a different context (for example, Hamiltonian motion in a potential on the surface of an hyperboloid, rather than a sphere); they have the advantage of being entirely expressible in terms of elementary functions.

As indicated in §5.1, the Weierstrass function $W(u)$ introduced in §4.3 (equation (4.31)) degenerates to trigonometric when the parameter takes the value $m = 1$ (i.e. $\hat{E} \equiv \frac{9}{2}m = \frac{9}{2}$), namely

$$W = \tfrac{1}{3} + \tfrac{1}{2}w^2, \tag{D 1}$$

where

$$w = \tan(u/\sqrt{2}) \tag{D 2}$$

up to arbitrary translations of the independent variable u. Hence we expect that the functions $U(u)$, $V(u)$ (which represent the evolution of the ellipsoidal cloud's shape) will be expressible rationally in terms of $w \equiv \tan(u/\sqrt{2})$.

When $m = 1$, the cubic $N(\alpha, \beta)$ (see (4.11)) becomes decomposable (cf. (5.2)):

$$N(\alpha,\beta) \equiv (\alpha+\beta-1)(\alpha^2-\alpha\beta+\beta^2+\alpha+\beta+1) \tag{D 3}$$

and there are thus two cases to consider: we shall treat the simpler one:

$$(\alpha+\beta-1) = 0. \tag{D 4}$$

(The curve corresponding to the vanishing of the second factor in (D 3) is entirely complex, except for an isolated real point: $\alpha = -1 = \beta$.) Then, according to (4.10)

$$V^2 = \alpha U^2 + (1-\alpha). \tag{D 5}$$

The spherical trajectories described by (D 5) may be parametrized by means of the parameter $\tau \equiv (V-1)/(U-1)$, as

$$U = \frac{\tau^2 - 2\tau + \alpha}{\tau^2 - \alpha}, \quad V = \frac{-(\tau^2 - 2\alpha\tau + \alpha)}{\tau^2 - \alpha}. \tag{D 6}$$

We thus have

$$\frac{\mathrm{d}U}{\mathrm{d}\tau} = \frac{-2V}{\tau^2-\alpha}; \quad \frac{\mathrm{d}V}{\mathrm{d}\tau} = \frac{-2\alpha U}{\tau^2-\alpha}$$

and, following the method of §4.3, it is seen that the function $\tau(u)$ obeys the differential equation

$$\tau'^2(u) = \frac{2}{3\alpha(1-\alpha)}(\tau-\alpha)^2[\tau^2 + 2\tau(\alpha-1) - \alpha] \tag{D 7}$$

which may be solved in the form

$$(\tau-\alpha) + (\tau^2 + 2\tau(\alpha-1) - \alpha)^{1/2} = S(u), \tag{D 8}$$

where $S(u)$ is proportional to $w(u)$:

$$S(u) \equiv (3\alpha(1-\alpha))^{1/2}\, w(u). \tag{D 9}$$

This expression is readily solved for the unknown $\tau(u)$:

$$2\tau = \frac{S^2 + 2\alpha S + \alpha(\alpha+1)}{S + (2\alpha - 1)}, \tag{D 10}$$

as expected, τ admits a rational expression in terms of $S(u)$ or, equivalently, $w(u)$. Rational expressions for the unknowns U and V may then be deduced through equation (D 6), namely

$$U = \frac{S^4+4S^3(\alpha-1)+2S^2(\alpha-1)(3\alpha-2)+4\alpha S(\alpha^2-1)+\alpha(\alpha-1)(\alpha^2+11\alpha-8)}{S^4+4\alpha S^3+2\alpha S^2(3\alpha-1)+4\alpha S(\alpha-1)(\alpha-2)+\alpha(\alpha-1)(\alpha^2-13\alpha+4)} \quad \text{(D 11)}$$

and a similar expression for V.

We remark that we still have the freedom to perform an arbitrary translation of $u: u \to u+u_0$, and that induces a homographic transformation on the function $S(u) \equiv [3\alpha(1-\alpha)]^{1/2}\tan(u/\sqrt{2})$. The remarkable fact is that, through an appropriate choice of the translation parameter u_0, the expressions for both U and V can be made *even* in the variable S, and may accordingly be simplified.

We introduce a new parameter h in place of α, in order to have a fully rational result:

$$\alpha \equiv h(h+2)/(h^2-1) \quad \text{(D 12)}$$

and obtain

$$U = \frac{9w^4h^2(h+1)+6w^2(h^2-1)(2h+1)+(h+2)^2(h-1)(2h+1)}{9w^4h(h+1)+6w^2h(h+2)(1-h^2)+(2h+1)^2(h+2)(1-h)}, \quad \text{(D 13)}$$

together with an expression of the same kind for V:

$$V = \frac{-9w^4h(h+1)^2-6w^2h(h+2)(2h+1)+(h-1)^2(h+2)(2h+1)}{9w^4h(h+1)+6w^2h(h+2)(1-h^2)+(2h+1)^2(h+2)(1-h)}. \quad \text{(D 14)}$$

Appendix E. An invariant measure of oblateness of tri-axial ellipsoids

The shape of a tri-axial ellipsoid of axes a, b, c is fixed by the two ratios $H \equiv b/a$, $K \equiv c/a$; these two quantities however may not directly be interpreted as representing the shape of an ellipsoid, not being invariant under relabelling of the three axes. Two related quantities may be constructed from H and K, which have the property of being invariant:

$$\frac{\delta}{UV} \equiv \frac{1+U^3+V^3}{UV}, \quad \text{(E 1)}$$

and the product

$$\frac{(U+V-2)(U-2V+1)(V-2U+1)}{UV}, \quad \text{(E 2)}$$

where, as always in the present work, $U \equiv H^{2/3}$, $V \equiv K^{2/3}$.

Let us form the combination of these two invariants, denoted ζ:

$$\zeta \equiv \frac{(U+V-2)(U-2V+1)(V-2U+1)}{3UV-\delta}. \quad \text{(E 3)}$$

For an axisymmetric ellipsoid – e.g. one with $V = 1$ – ζ reduces to

$$\zeta[U; V=1] \equiv 2\frac{U-1}{U+2} \quad \text{(E 4)}$$

(a monotonic function of U) and is a measure of the oblateness of the ellipsoid: for a flat disk, $U = 0$ and hence $\zeta = -1$; for a sphere, $U = 1$ and hence $\zeta = 0$; for an elongated (cigar-shaped) ellipsoid, $U \to \infty$ and $\zeta \to 2$.

Thus, $0 < \zeta < 2$ for prolate spheroids, and $-1 < \zeta < 0$ for oblate ones. For a general triaxial ellipsoid, we may retain the invariant ζ as a definition of its oblateness.

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